Phase synchronization of chaotic systems with small phase diffusion

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The geometric theory of phase locking between periodic oscillators is extended to phase coherent chaotic systems. This approach explains the qualitative features of phase locked chaotic systems and provides an analytical tool for a quantitative description of the phase locked states. Moreover, this geometric viewpoint allows us to identify obstructions to phase locking even in systems with negligible phase diffusion, and to provide sufficient conditions for phase locking to occur. We apply these techniques to the Rössler system and a phase coherent electronic circuit and find that numerical results and experiments agree well with theoretical predictions.

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I. INTRODUCTION

While the study of phase locking between periodically oscillating systems dates back to Huygens [1] the investigation of phase locking between chaotic systems has a more recent history. Its occurrence was noted in [2–4] and the phenomenon has since been observed in such diverse systems as electrically coupled neurons [5,6], spatially extended ecological systems [7], earthquake models [8], a plasma discharge tube [9], and its potential role in brain functions has been recognized [10,11].

Although much work has been done on detecting and analyzing chaotic phase synchronization (CPS), the phenomenon is still not completely understood and predictive methods are still lacking. In the chaotic systems studied in [4] it is possible to define a phase variable which varies periodically up to a small chaotic term. If this chaotic term can be treated as white noise then the theory developed in [12] is applicable. A similar approach is considered in [13,14] where the phase is modeled by a stochastically driven overdamped particle. Since periodic orbits form a skeleton of a chaotic attractor, it was argued in [15] that CPS can be described in terms of the phase locking properties of these periodic orbits. The detailed structure of attractors in the CPS regime was analyzed further in [16].

Many of these approaches describe behavior that agrees well with that observed in systems exhibiting CPS. However, predictive methods for computing when and how CPS occurs have not been discussed in detail. Moreover, the question whether CPS is possible in all phase coherent systems has, to our knowledge, not been addressed.

In the following we describe such predictive methods based only on information about the unperturbed system, the type of driving signal and the nature of the coupling. We give a geometrical description of how CPS occurs, and show how to predict the driving strength necessary for CPS, the phase difference between the drive and response in the CPS regime, and by how much this phase difference varies. Moreover, this approach also identifies a geometric obstruction to CPS in phase coherent systems, which may be present even if the phase diffusion is negligible. This theory is well developed in the case of systems with stable limit cycles [17–20] and we address the questions when and how it can be extended to systems with stable, phase-coherent chaotic attractors.

In Sec. II we give the mathematical details of how the geometric theory of periodic phase locking described in [20] can be extended to chaotic oscillators. These ideas are applied to the Rössler equations (Sec. III) and to an electronic circuit based on the partially linear Rössler equations (Sec. IV), and the theoretical predictions are verified numerically and experimentally. In Sec. V we further discuss sufficient conditions under which a phase-coherent attractor can be phase locked to a periodic drive, and demonstrate that the amount of phase diffusion and the geometry of the attractor are equally important. Sufficient conditions under which a chaotic attractor is phase-coherent are discussed in the Appendix.

II. A DESCRIPTION OF CPS USING ISOCHRONS

A frequent goal in science and engineering is to predict how the behavior of a periodically oscillating system changes when it is subject to an outside perturbation [17,19]. Ideally such predictions should be based only on information about the unperturbed system, and the type of perturbation acting upon it, thus avoiding the work of performing numerous experiments. In this section we describe how this can be achieved in the case of a small, periodic perturbation acting on a chaotically oscillating system.

We first review the theory for a periodically perturbed nonlinear oscillator following the approach in [20]. Assume that the system $\mathbf{X}'=\mathbf{F}(\mathbf{X})$ has an exponentially stable limit cycle ρ of period T. It is possible to find coordinates $(\phi, \mathbf{R})$ in a tubular neighborhood N of ρ such that the phase ϕ is the angular distance along ρ, $\mathbf{R}$ measures the radial distance from ρ, and $\phi' = d\phi/dt = 1$.

The level sets of ϕ are called *isochrons* and define codimension one manifolds that foliate N. Every isochron intersects ρ in a single point q_ϕ, called the *basepoint* of the isochron. Let φ_t be the flow of $\mathbf{X}'=\mathbf{F}(\mathbf{X})$. There exist $C,k>0$ such that for any point p on an isochron with basepoint q_ϕ we have $|\varphi_t(p)-\varphi_t(q_\phi)| \leqslant Ce^{-kt}$. Therefore the asymptotic behavior of all points on an isochron is the same as that of its basepoint.

If the system is subject to a small perturbation $\epsilon\mathbf{p}(t)$ of period T_d so that $\mathbf{X}'=\mathbf{F}(\mathbf{X})+\epsilon\mathbf{p}(t)$, the theory of invariant manifolds [21] implies that the perturbed system possesses an attracting limit cycle ρ_ϵ which is $O(\epsilon)$ close to ρ. A direct calculation shows that

$$\begin{aligned}\phi' &= 1+\epsilon\nabla_{\mathbf{X}}\phi|_{\rho_\epsilon(\phi)}\cdot\mathbf{p}(t)\\ &= 1+\epsilon\nabla_{\mathbf{X}}\phi|_{\rho(\phi)}\cdot\mathbf{p}(t)+\mathcal{O}(\epsilon^2)\\ &\overset{\text{def}}{=} 1+\epsilon\Omega(\phi,t)+\mathcal{O}(\epsilon^2),\end{aligned}$$

where $\nabla_{\mathbf{X}}\phi|_{\rho_\epsilon(\phi)}$ is the gradient of $\phi(\mathbf{X})$ evaluated at the point $\rho_\epsilon(\phi)$ of the perturbed orbit and the first equality follows from the $O(\epsilon)$ closeness of ρ and ρ_ϵ. Since $\nabla_{\mathbf{X}}\phi$ points along the direction of fastest increase of ϕ, it can be interpreted as the phase-dependent sensitivity of ϕ, and so $\Omega(\phi,t)=\nabla_{\mathbf{X}}\phi|_{\rho(\phi)}\cdot\mathbf{p}(t)$ measures the influence of the perturbation on the phase.

Defining the phase difference between $\mathbf{p}(t)$ and ϕ as $\Psi=\phi-(T/T_d)t$ and letting $\epsilon\Delta=1-T/T_d$, we obtain to second order in ϵ

$$\Psi'=\epsilon\left[\Delta+\Omega\left(\frac{T}{T_d}t+\Psi,t\right)\right]. \tag{1}$$

Averaging this equation over one period of the drive gives

$$\Psi'=\epsilon[\Delta+\Gamma(\Psi)], \tag{2}$$

where the function $\Gamma(\Psi)$ is the average

$$\Gamma(\Psi)=(1/T_d)\int_0^{T_d}\Omega\left(\frac{T}{T_d}t+\Psi,t\right)dt. \tag{3}$$

If this equation has a stable fixed point Ψ_0, then the phase ϕ approaches the solution $\phi(t)=\Psi_0+(T/T_d)t$, so that $\phi(t+T_d)=\phi(t)$ and the system is phase locked with the drive with a phase difference of Ψ_0.

To extend these ideas to the case of chaotic systems we assume that $\mathbf{X}'=\mathbf{F}(\mathbf{X})$ possesses a chaotic attractor A and that there exist coordinates ($\mathbf{R}$, ϕ) in a neighborhood of A such that

$$\mathbf{R}'=\mathbf{F}(\mathbf{R},\phi), \tag{4}$$

$$\phi'=1+\delta(\mathbf{R},\phi), \tag{5}$$

where ϕ is T periodic. We require that $\delta(\mathbf{R},\phi)$ is $O(\eta)$ where $\eta\ll 1$ except possibly for ϕ in a set of total length $O(\eta)$ on which $\delta(\mathbf{R},\phi)$ can be $O(1)$, or, equivalently, that $\int_0^T\delta(\mathbf{R},\phi)d\phi=O(\eta)$ for any orbit on A. It follows that ϕ completes one period in time $T+O(\eta)$. Moreover, two points (R_1,ϕ) and (R_2,ϕ) sharing the same initial phase will remain close in phase for times at least $O(1/\eta)$ before they are separated by the effects of the term δ. Therefore the level sets of ϕ form approximate isochrons and the system may be called *phase-coherent* [22].

It is not always clear when such a change of coordinates exists. However, given a system of differential equations, or a timeseries it is frequently possible to define a phase Φ and a natural period T such that $|\Phi(T)-\Phi(0)|<\eta\ll 1$. This can be done using the Hilbert transform, or other approaches [23–25]. In the Appendix we show that if in addition Φ is strictly increasing, then there exists a change of coordinates for which Eqs. (4) and (5) hold.

The dispersion of the phase due to the term δ is frequently referred to as *phase diffusion* since the effect is similar to that of a random perturbation of a periodically forced phase oscillator [12]. Let us emphasize that δ does not necessarily behave like δ-function correlated white noise. The correlation $\langle\delta(t),\delta(t+\tau)\rangle_t$ may, in general, decay relatively slowly with τ, and thus the theory developed in [12] is not necessarily valid for CPS. In the following we investigate phase synchronization when δ is small in the sense described above. In particular, it is not necessary that the system is chaotic, as long as there are coordinates in which the system is given by Eqs. (4) and (5). The coordinates ($\mathbf{R}$, ϕ) are not uniquely defined but may be chosen in any way such that δ satisfies the conditions given above.

As in the periodic case, we want to predict the response of the phase to a small periodic perturbation by analyzing the system $\mathbf{X}'=\mathbf{F}(\mathbf{X})+\epsilon\mathbf{p}(t)$. We assume that the original system is stable to small perturbations in the sense that the perturbed system has an attractor A_ϵ which is close to the attractor of the unperturbed system in the sense that a typical orbit on A has a counterpart on A_ϵ and the two stay close over one oscillation. In particular, we do not assume that the dynamics on the two attractors is conjugate as, for instance, one of A_ϵ and A could be a chaotic and the other a periodic attractor.

Since $\delta(\mathbf{R},\phi)$ is continuous in both arguments, it will remain small when evaluated along an orbit of the new attractor A_ϵ. The same calculations as in the periodic case yield

$$\phi'=1+\delta(\mathbf{R},\phi)+\epsilon\nabla_{\mathbf{X}}\phi|_{(\mathbf{R},\phi)}\cdot\mathbf{p}(t), \tag{6}$$

where $\nabla_{\mathbf{X}}\phi|_{(\mathbf{R},\phi)}$ is the gradient of $\phi(\mathbf{X})$ at a point ($\mathbf{R}$, ϕ).

We will assume that $\nabla_{\mathbf{X}}\phi|_{(\mathbf{R},\phi)}$ satisfies $\nabla_{\mathbf{X}}\phi|_{(\mathbf{R}_1,\phi)}=\nabla_{\mathbf{X}}\phi|_{(\mathbf{R}_2,\phi)}+O(\epsilon)$ for all pairs of points $(\mathbf{R}_1,\phi),(\mathbf{R}_2,\phi)$ in a neighborhood of A so that the phase dependent sensitivity is constant as a function of $\mathbf{R}$ up to terms of order ϵ. This is a strong assumption which is approximately satisfied for the systems described in Secs. III and IV, and whose necessity will be discussed further in Sec. V.

Using the same definitions as in the periodic case, we now find up to second order in ϵ:

$$\Psi'=\epsilon\left[\Delta+\Omega\left(\frac{T}{T_d}+\Psi,t\right)\right]+\delta(\mathbf{R},\phi). \tag{7}$$

Again, we proceed by averaging the term in the brackets of Eq. (7). Following the usual proof of the Averaging Theorem [26] we introduce the near identity transformation

$$\Psi=\tilde{\Psi}+\epsilon u(\tilde{\Psi},t),$$

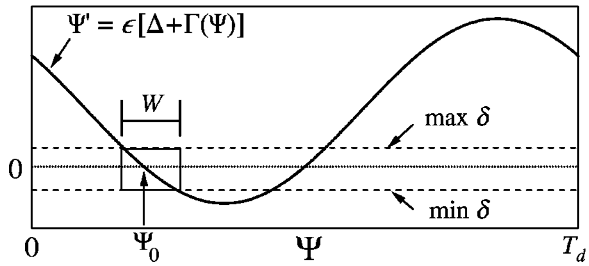

FIG. 1. Schematic representation of Eq. (8) for Ψ'. Once in the interval *W*, the relative phase Ψ cannot escape. The value of Ψ_0 estimates the phase difference between the drive and the system response, and the size of *W* estimates the variation in this difference.

where $u(\tilde{\Psi},t)=\Omega(\tilde{\Psi},t)-\Gamma(\tilde{\Psi})$ and $\Gamma(\tilde{\Psi})$ is defined as in Eq. (2). It follows that up to second order in ϵ

$$\tilde{\Psi}'=\epsilon[\Delta+\Gamma(\tilde{\Psi})]+\tilde{\delta}(\mathbf{R},\phi), \tag{8}$$

where $\tilde{\delta}(\mathbf{R},\phi)=\delta(\mathbf{R},\phi)-\epsilon\delta(\mathbf{R},\phi)\partial_{\tilde{\Psi}}u(\tilde{\Psi},t)$. If δ is small for all values of ϕ then Eq. (8) is of form (2), with a small perturbation $\tilde{\delta}$ whose exact nature depends on the driven system.

Since both Ψ and $\tilde{\Psi}$ and δ and $\tilde{\delta}$ are uniformly close, we will drop the tildes from now on. Define the region

$$\mathcal{W}\stackrel{\text{def}}{=}\{\Psi:\min\delta<\epsilon[\Delta+\Gamma(\Psi)]<\max\delta\}.$$

For ϵ sufficiently large, $\mathcal{W}$ is a proper subset of $[0,T_d]$. If Ψ_0 is a stable fixed point of Eq. (2) and *W* the component of $\mathcal{W}$ containing Ψ_0, a Lyapunov function argument shows that *W* is a stable inflowing region for Eq. (8). This is shown schematically in Fig. 1.

A similar argument holds if $\delta(\mathbf{R},\phi)$ is of $O(1)$ during a time of $O(\eta)$ in the cycle. If $\eta=O(\epsilon)$ then for any initial relative phase $\Psi(0)$ in Eq. (8) we have $\Psi(t)=\Psi(0)+O(\epsilon)$ for $0\leqslant t\leqslant T_d$. Therefore during one period of the drive $\Gamma(\Psi(t))=\Gamma(\Psi(0))+O(\epsilon)$ and Ψ advances by

$$\Psi(T_d)-\Psi(0)=\epsilon\int_0^{T_d}[\Gamma(\Psi(0))+\delta(\mathbf{R},\phi)]dt+\mathcal{O}(\epsilon^2), \tag{9}$$

and for sufficiently large ϵ points starting in the basin of attraction of a stable fixed point Ψ_0 of Eq. (2) are still attracted to the vicinity of Ψ_0 in Eq. (8).

Note that in obtaining Eq. (8) we have only averaged the periodic term Ω, but not δ. Alternatively we can also average $\delta(\mathbf{R},\phi)$ over a time $nT_d\ll 1/\epsilon$ to obtain $\Psi'=\epsilon[\Delta+\Gamma(\Psi)+\overline{\delta}(t)]$ up to second order in ϵ where $\overline{\delta}(t)=1/(nT_d)\int_0^{nT_d}[\delta(\mathbf{R}(t),\phi(t))]dt$. Since $\delta(\mathbf{R},\phi)$ is not periodic in time, the solution of this equation will in general be a good approximation to the unaveraged solution only for times up to order $1/\epsilon$ [27], while the approach outlined above gives results valid for all time. However, we can still recover the conclusions of the above argument as follows: Suppose that we replace δ by $\overline{\delta}$ in Eq. (8) and there exists a trapping region *W* for the phase difference Ψ. Then any solution of the unaveraged equation starting in *W* will stay in this wedge for times up to $O(1/\epsilon)$. Therefore we can string together infinitely many intervals over which this solution is valid to show that *W* is a trapping region of the unaveraged equation for all time. Although the solutions of the averaged and unaveraged equations may not remain close, we can still conclude that both will remain trapped around Ψ_0 for all time. It is therefore a matter of convenience whether we choose to average $\delta(\mathbf{R},\phi)$ or not. Since averaging $\delta(\mathbf{R},\phi)$ is justified, average quantities such as the phase diffusion constant are meaningful in estimating the size of δ even when δ is different from white noise, as long as the average is approached quickly compared to $1/\epsilon$.

For periodic systems the transition to phase locking occurs as follows: As ϵ increases, the graph of $\epsilon[\Delta+\Gamma(\Psi)]$ is dilated vertically (see Fig. 1). System (2) nears a saddle-node bifurcation and Ψ spends more time in the vicinity of the incipient bifurcation point. At a critical value of ϵ, Eq. (2) undergoes a saddle-node bifurcation, giving birth to a stable fixed point Ψ_0. At this point the system enters the 1:1 Arnold tongue and phase locks to the drive. The transition to CPS in the perturbed system (7) is similar, but more gradual. Even as the saddle-node bifurcation gives rise to a stable point Ψ_0 of Eq. (2), the term δ in Eq. (7) may cause the phase to slip out of a neighborhood of Ψ_0. If $T_d>T$, then $\epsilon\Delta>0$ and the graph of $\Gamma(\Psi)$ is shifted upwards. In this case δ typically causes a forward slip in the phase. If $T_d<T$, the opposite is true. As ϵ grows, these slips become rarer and disappear altogether with the creation of a trapping region *W* for the phase. If max δ and min δ remain approximately constant as ϵ is increased, then the region *W* moves and becomes narrower, and phase locking typically becomes tighter. This process is illustrated in Secs. III and IV, and in Fig. 2.

It will be shown in Sec. V that it is frequently important to assume that $\nabla_X\phi|_{(\mathbf{R},\phi)}$ is approximately constant in $\mathbf{R}$. We also note that the existence of the region *W* is a sufficient, but not necessary condition for CPS. The attraction to an approximately phase locked state may become stronger than the phase diffusion even before the coupling is sufficiently strong for the region *W* to appear. This depends on the particular form of the term δ, and without further assumptions it is difficult to say more.

The term $\delta(\mathbf{R},\phi)$ is a *deterministic* noise term. More information about this term leads to additional information about the phase locked state. In [24,28] it is argued that δ can typically be approximated by fractional Brownian motion. Intermittent spikes in δ may lead to intermittent phase slips [29]. Statistical information about such spikes yields direct information about the frequency of phase slips in such systems [30]. Note that we have only assumed that δ is small and not that it is modeled well by any particular stochastic process.

In this argument it was assumed that the periodic driving has only a small effect on the attractor *A* and that ϵ is small enough to justify averaging. However, ϵ needs to be sufficiently large for a phase trapping region *W* to appear. These

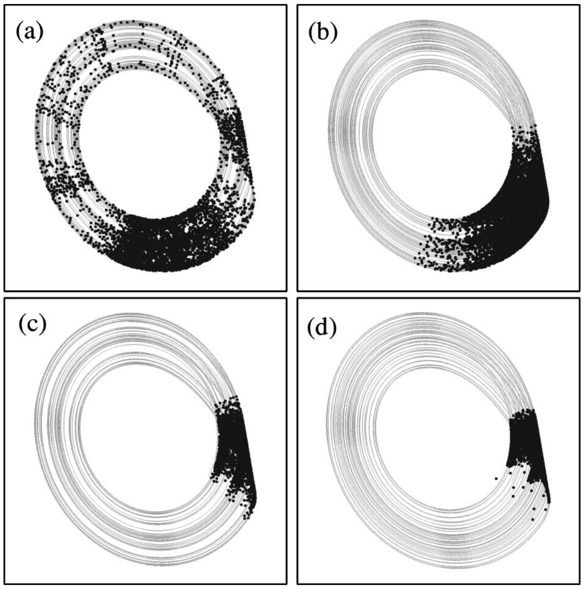

FIG. 2. Numerical simulations of system (11) for $\gamma=0.10$, driven in y using $\epsilon \sin(\omega_d t)$ with $\omega_d=0.711$. In (a)–(d), $\epsilon=0.002$, 0.005, 0.02, 0.05 and the region plotted in each panel is given by $-4.3 \leqslant x \leqslant 4.3$ and $-5.8 \leqslant y \leqslant 5.1$. The dark points show the Poincaré section at zero drive phase. For $\epsilon=0$ (data not shown), the points are distributed relatively evenly over the attractor. In (a), ϵ is small and the points become concentrated near $\Psi=-\pi/2$, but frequent phase slip events are still evident. As ϵ increases, these events become less frequent and eventually a phase locking region appears (b). For still larger ϵ, the region moves towards $\Psi=0$ (c) and becomes narrower (d). For larger γ (data not shown), δ is larger and the trapping region is correspondingly broader. When the drive is applied to the x variable (data not shown), $\Psi \approx -\pi$ at the threshold of locking, and Ψ approaches $-\pi/2$ for large ϵ.

two opposing conditions on ϵ may not always be compatible. It is therefore necessary to treat phase-coherent attractors case by case. Fortunately perturbation results often hold for a wide range of values of ϵ, and we therefore expect that these ideas are widely applicable.

III. THE RÖSSLER SYSTEM

In this section we consider the Rössler system with a periodic drive in the x variable [31]

$$x' = -y - z + \epsilon \sin \omega_d t,$$

$$y' = x + by, \tag{10}$$

$$z' = 0.2 + z(x-10),$$

with $b=0.12$. Introducing cylindrical coordinates $r=\sqrt{x^2+y^2}$, $\theta=\arctan(y/x)$ and z we obtain the following equations:

$$\theta' = 1 + \frac{z \sin \theta}{r} + \frac{b}{2}\sin 2\theta,$$

$$r' = z \cos \theta + br \sin^2 \theta.$$

The right-hand side of the phase equation contains two terms that cause its velocity to deviate from constant: $S(t)=z \sin \theta/r$ and $(b/2)\sin 2\theta$. Since the second term is periodic in θ and does not depend on z and r its contributions in the following calculations are orders of magnitude smaller than those of the first term, and we do not consider it further. The variable $z(t)$ is close to zero, except for a sequence of spikes that always occur near $\theta=\theta^* \approx 0.22\times\pi$. During these spikes the phase velocity increases causing phase jumps and an increase in the average angular frequency to $\omega=1.0329$.

As explained in the previous section, we would like to define a new phase coordinate ϕ in a neighborhood of the attractor such that $\phi'=1+\delta(t)$ where $\delta(t)$ is small on average and has zero mean. To do this we model $S(t)$ by a sequence of constant pulses $\pi(\theta)$ around the phase value $\theta=\theta^*$ which are defined as $\pi(\theta)=\lambda$ if $\alpha \leqslant \theta < \beta$ and is $\pi(\theta)=0$ otherwise. We choose the values $\alpha=0.18\pi$, $\beta=0.28\pi$, and $\lambda=1.74$ so that the period of a phase variable with phase velocity $1+\pi(\theta)$ is the average period of the Rössler system $T=6.0838$.

The new phase is now defined by $\phi'=\theta'/(1+\pi(\theta))$ and $\phi(0)=\theta(0)=0$, so that ϕ is periodic with period $2\pi+(\alpha-\beta)\lambda/(\lambda+1)=T$ and satisfies

$$\phi' = \frac{1+z \sin \theta(\phi)}{r(1+\pi[\theta(\phi)])} = 1 + \delta(r,z,\phi),$$

where $\delta(r,z,\phi)$ has zero mean and is large during only a small fraction of each cycle.

The phase perturbation term Ω is

$$\Omega(\phi,t) = -\frac{1}{r(1+\pi[\theta(\phi)])}\sin\theta(\phi)\sin\omega_d t$$

and the function $\Gamma(\Psi)$ is computed by averaging $\Omega(\omega_d t/\omega, t)$, as in Eq. (3). The exact result is complicated, but $\Gamma(\Psi) \approx -\cos(\Psi)/2r$ is a good approximation. Since r is not constant we use its average value in the approximations below. Details of the calculation may be found at the author's web site [32].

The value of ϵ necessary to phase lock the system when $\omega_d=\omega$ is less than 0.001. Since this value is an order of magnitude smaller than the coupling values considered below phase diffusion has a negligible effect in this case. We also estimate $\delta(r,z,\phi)$ by noting that $\phi(nT)-\phi(0)=\int_0^{nT}\delta(r,z,\phi)$. Thus the average $\bar{\delta}$ can be estimated from computing $(\phi(nT)-\phi(0))/(nT)$ over many orbits. The maximum value of $\bar{\delta}$ with $n=1000$ is of order 10^{-4} and it therefore has no significant influence. The phase diffusion coefficient computed as in [4,16] equals $D_\phi=8.28\times10^{-6}$.

Interestingly $\delta(r,z,\phi)$ is relatively large over most oscillations. However, computing the average of δ over one oscillation $\bar{\delta}(t)=1/T\int_0^T\delta(r,z,\phi)dt$ we find that $\bar{\delta}(nT)$ and $\bar{\delta}((n+1)T)$ are strongly negatively correlated. Thus most forward jumps in ϕ are followed by a backward jump which leads to the small value of the phase diffusion. Since we can

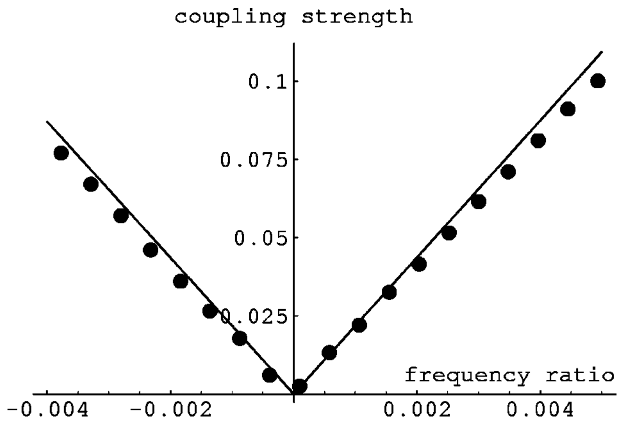

FIG. 3. The 1:1 Arnold tongue for the Rössler equations. The values ϵ and $\epsilon\Delta$ are plotted on the vertical and horizontal axis, respectively. The lines represent the theoretical prediction, while the dots are obtained from numerical simulations.

use averages to estimate the influence of δ it follows that it has a negligible influence despite its large values during each oscillation.

The Arnold tongue is computed by finding the minimal values of the coupling strength $\epsilon_{\min}$ at which phase locking occurs for a given value of $\epsilon\Delta = 1-\omega_d/\omega$, where $\omega = 2\pi/T$. In particular we find the value Ψ_0 at which $\Gamma(\Psi)$ reaches its maximum if $1-\omega_d/\omega<0$, or minimum if $1-\omega_d/\omega>0$ so that $\epsilon_{\min}$ can be found from Eq. (8) as

$$\epsilon_{\min}=\frac{\omega_d/\omega-1}{\Gamma(\Psi_0)},$$

where we have neglected the term δ. The value of Ψ_0 gives the phase difference at which the phase locking first occurs. The results of this approximation are compared with data from numerical simulations in Fig. 3. We also find that $\Psi_0 \approx -\pi$ when $\epsilon\Delta>0$ and $\Psi_0 \approx \pi$ when $\epsilon\Delta<0$ which also agrees well with the numerical simulations. We have also repeated the analysis with different types of periodic driving in Eq. (10) and again obtained good agreement between theory and numerical simulations.

It is interesting to note that our predictions overestimate the value of ϵ at which phase locking first occurs by about 10%. This is in part due to the use of the average value of r in $\Gamma(\Psi)\approx -\cos(\Psi)/2r$ which makes our approximation of $\Gamma(\Psi)$ independent of r. The dynamics of r, which are ignored in this approximation, may play an important role in determining phase locking as demonstrated in Sec. V. A more careful analysis can improve these predictions, but is beyond the scope of this paper.

IV. APPLICATION TO EXPERIMENTS

To experimentally confirm the analysis above, we constructed a phase-coherent chaotic electronic circuit modeled by the following equations:

$$x' = -\alpha(x/20+y/2+z),$$

$$y' = -\alpha(-x-\gamma y)+\epsilon \sin \omega_d, \qquad (11)$$

$$z' = -\alpha[-15(x-3)\theta(x-3)+z],$$

where $\theta(x)$ is the step function and $\alpha=10^4$ sets the experimental time scale. System (11) can be viewed as a piecewise linear approximation of the Rössler system discussed in the previous section. It has been used in previous studies of chaotic synchronization [33]. For experimental circuit details, see [34].

The phase space of Eq. (11) is divided into two regions, $R_1=\{(x,y,z)\in\mathbb{R}^3 : x<3\}$ and $R_2=\mathbb{R}^3-R_1$, in each of which the equations are linear. By changing coordinates so that the system is in normal form in R_1, the solutions of Eq. (11) in R_1 have the form $(w(t),z(t))=e^{(\nu+i\omega)t}w(0)+e^{-t}z(0)$, where $w(t)=x(t)+iy(t)$. In R_1 the solutions approach the xy plane, which is invariant. If $\gamma>0.05$, then $\nu>0$ and the origin is a spiral source in the xy plane. The parameter γ controls the instability of the origin since ν increases with increasing γ.

When an orbit enters R_2, it is lifted off the xy plane. Shortly thereafter, the orbit is reinjected into R_1 closer to the z axis, it quickly approaches the xy plane and, if $\nu>0$, spirals outwards until it reenters R_2 and the process repeats. It can be shown that this behavior results in a Poincaré return map similar to the Hénon map [35].

We now define a phase coordinate as

$$\phi=(\omega r)^{-1}\arctan(y/x), \qquad (12)$$

where r is the average attractor radius, which depends on ν. It follows that in R_1 the sets $I_c=\{\phi:\phi=c\}$ form an invariant family, $\phi'=1$, and $\nabla_{\mathbf{x}}\phi$ is constant on each I_c. These observations permit a straightforward calculation of $\Gamma(\Psi)$ as in the previous section. Since all orbits eventually enter the region R_2, this description of the phase is incomplete. However, the size of the errors in this approximation depends directly on the size and frequency of the excursions into the region R_2. These in turn depend on ν, which can be directly controlled in experiments via the parameter γ, which has a similar effect on the dynamics as the parameter b in the Rössler system. This allows us to adjust the magnitude of δ in Eq. (8).

Numerical and physical experiments were conducted by changing the magnitude of the driving term $\epsilon\sin(\omega_d t)$ in Eq. (11). Using Eq. (12) and the ideas of Sec. II, in normal coordinates we obtain

$$\Gamma(\Psi)=0.021\cos(\omega\Psi)-0.1666\sin(\omega\Psi) \qquad (13)$$

for $\gamma=0.127$ and $r=5.12$. Returning to the original coordinates of system (11), we see that if the frequency of the drive ω_d is larger than the intrinsic frequency of the circuit ω_0 [36], i.e., $T_d<T$ and $\epsilon\Delta<0$, we expect that the circuit first locks to the drive with a phase difference $\Psi\approx-\pi/2$ and that Ψ moves towards 0 as ϵ is increased. Similarly, if $\omega_d<\omega_0$ we expect that initially $\Psi\approx\pi/2$, and Ψ moves towards 0 as ϵ is increased. The theoretical analysis above yields good qualitative (Figs. 2 and 4) and quantitative agreement with the experimental data in the location, size, and shape of the

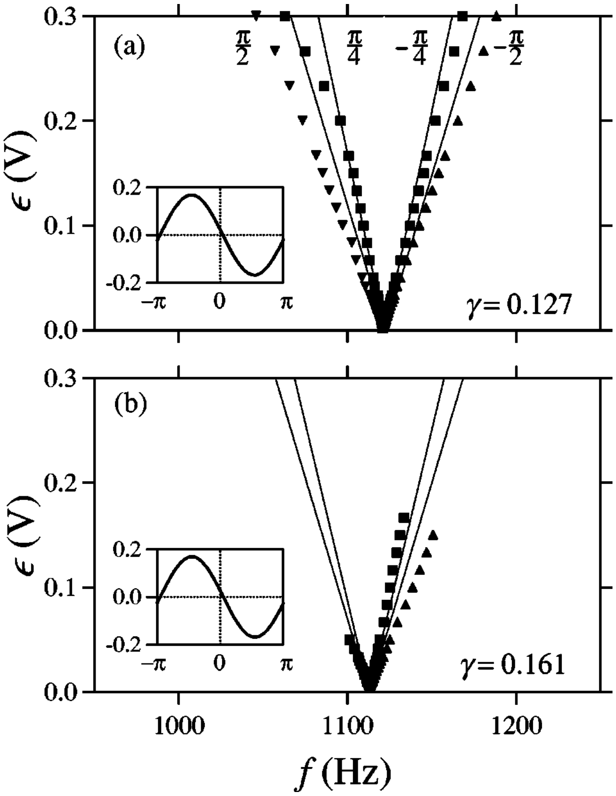

FIG. 4. CPS phase locking results from experiment (symbols) and theoretical analysis (lines) for system (11) for $\gamma = 0.127$ (a) and $\gamma = 0.161$ (b). The system is periodically driven in y with frequency and amplitude as shown on the axes. The average frequencies of the undriven system are 1122 Hz and 1113 Hz, corresponding to $\omega_0 = 0.705$ and 0.699, in (a) and (b). Triangles indicate when the system lies just at the threshold of slipping, while squares indicate parameters for which $|y| \approx |x|$ and $\Psi \approx \pm \pi/4$, as indicated at the top of (a). The wedge-shaped regions are analogous to Arnold tongues in the periodic case. The lines are calculated from $\Gamma(\Psi)$ and Eq. (2). Insets: $\Gamma(\Psi)$ vs Ψ, as obtained from Eq. (13) for (a). For (b) the coefficients of the terms in Eq. (13) are 0.025 and -0.1666.

phase-locked region (Fig. 4) indicating that the approximations in the phase descriptions were appropriate. We repeated the analysis with different types of driving and obtained similar agreement between theoretical prediction and experimental results.

Note that the analytical predictions again overestimate the size of the coupling strength ϵ necessary for phase locking, as in the previous case. This is not surprising as the dynamics of systems (11) and (10) are similar. Within the Arnold tongue, the circuit oscillates chaotically, but remains phase locked to the drive [37]. For large ϵ, the drive may be so strong that it imposes periodic dynamics on the circuit [38]. This occurs at the top of Fig. 4(b). We plot only the points for the region of CPS beneath this.

V. DEPENDENCE OF THE PHASE SENSITIVITY OF ϕ ON R

One of the main assumptions in Sec. II was that $\nabla_{\mathbf{X}}\phi|_{(\mathbf{R},\phi)}$ is approximately independent of $\mathbf{R}$. This assumption implies that the first two terms in Eq. (7) are independent of $\mathbf{R}$ and the calculations can proceed in a way similar to the periodic case.

This assumption is not necessarily satisfied. If the attractor A is chaotic, the section $A \cap \{\phi = c\}$ where $c \in [0,T)$ necessarily consists of infinitely many points. It is possible that $\nabla_{\mathbf{X}}\phi$ varies by a large amount on each such section even if $\delta(\mathbf{R},\phi)$ is small. To illustrate how this may happen we first present a transparent, although artificial example, and use a similar idea to construct a modification of the Rössler equations exhibiting a phase-coherent attractor which is difficult to phase lock.

Consider a planar vector field given in polar coordinates by $\mathbf{X}(r,\phi) = (0,1/r)$ in the annulus $r \in [1,2], \phi \in [0,2\pi]$. The isochrons are radial lines and the phase sensitivity $\nabla_{\mathbf{X}}\phi|_{(r,\phi)} = (-\cos\phi, \sin\phi)$ is independent of r. Let $\mathbf{Y} = S(r,\phi)\mathbf{X}$ where $S: \mathbb{R} \times S^1 \to \mathbb{R}$ is a unimodal function with a peak at $2(r-1)\pi$ and rapidly decaying to 1 away from this peak. The phase velocity of this system will be maximal at the angle $(r-1)\phi$ for the circular orbit at distance r from the origin and so $\nabla_{\mathbf{Y}}\phi$ depends on r. Applying the theory in Sec. II formally we see that symmetry implies that if the periodic orbit at $r=1$ is phase locked to a periodic drive with a phase difference Ψ_0, then the periodic orbit of radius r will phase lock to the same drive with a phase difference $\Psi_0 \pm 2\pi(r-1)$. Consider now an attracting orbit whose radius varies slowly between 1 and 2 and such that its phase velocity at each point is the same as that of $\mathbf{Y}$. Using the adiabatic approximation as in [39,40] we find that at each moment this orbit will be locked to the drive, although the phase difference between the two will vary between 0 and 2π. Although this example is artificial, it demonstrates that the dependence of $\nabla_{\mathbf{X}}\phi$ on $\mathbf{R}$ may be an intrinsic feature of a system with a significant influence on its phase locking characteristics.

To show what consequences this dependence may have on phase locking let $D(\mathbf{R},\phi) = \nabla_{\mathbf{X}}\phi|_{(\mathbf{R},\phi)}$ and let $D(\mathbf{R},\phi) = D_A(\phi) + D_V(\mathbf{R},\phi)$. Here $D_A(\phi)$ is the $\mathbf{R}$-independent part of $D(\mathbf{R},\phi)$, which can be obtained by averaging D over the attractor. The exact way of how $D_A(\phi)$ is obtained is unimportant for the following argument. From Eq. (6) it follows that

$$\Psi' = \epsilon[\Delta + D_A(\phi)\cdot\mathbf{p}(t)] + \epsilon D_V(\mathbf{R},\phi)\cdot\mathbf{p}(t) + \delta(\mathbf{R},\phi),$$

where now the term $\epsilon D_V(\mathbf{R},\phi)\cdot\mathbf{p}(t)$ is no longer assumed to be small compared to the term in the brackets. In particular, $D_V(\mathbf{R},\phi)$ varies by the same amount as $D(\mathbf{R},\phi)$ as a function of $\mathbf{R}$ regardless of the choice of $D_A(\phi)$. The approximate effect of this term on the phase over one oscillation can be calculated as in Eq. (9)

$$\Psi(T_d) - \Psi(0) = \epsilon \int_0^{T_d} D_V\left(\mathbf{R}, \frac{T}{T_d}t + \Psi\right)\cdot\mathbf{p}(t)\,dt. \quad (14)$$

As the effect of the term $\epsilon D_V(\mathbf{R},\phi)\cdot\mathbf{p}(t)$ increases with an increase in coupling strength it will not be necessarily possible to overcome it simply by increasing the coupling strength as in Sec. II. In particular, if D_V varies by a large amount from oscillation to oscillation or does not change sign, it is possible that it will prevent phase locking completely, regardless of the strength of the coupling. On the other hand, it is also possible that the average of $\epsilon D_V(\mathbf{R},\phi)\cdot\mathbf{p}(t)$ over a longer time interval nT_d, is small so that D_V has little effect on the phase difference and thus the arguments of Sec. II still apply. This is the case in Sec. III and IV. However, without more detailed information about the term $D_V(\mathbf{R},\phi)$ it is not possible to reach any of these conclusions even if $\delta(\mathbf{R},\phi)$ is small and the system is phase-coherent.

We illustrate this point using a modification of the Rössler equations in their polar form (11) with $b=0.14$. The equation for θ' is modified as follows:

$$\theta' = \left(\omega + \frac{z\sin\theta}{r} + \frac{b}{2}\sin 2\theta\right) g(r,\theta) + \epsilon \sin \omega_d t, \quad (15)$$

where the term $\epsilon\sin\omega_d t$ is a periodic drive and

$$g(r,\theta) = 1 - s\mathcal{N}(c(r-\alpha)\pi + \theta_0, \sigma^2),$$

where $\mathcal{N}(\mu,\sigma^2)=\exp[-(x-\mu)^2/(2\sigma^2)]$ is an unnormalized Gaussian-like function with a narrow peak μ whose width is determined by σ^2. The parameter $\alpha=7$ is set to the approximate inner radius of the $x-y$ projection of the Rössler attractor. In the simulations we have chosen $\theta_0=0.7\pi$, $c=7/24$, and $s=0.5$.

The effect of the term $g(r,\theta)$ is to slow down the phase variable θ whenever it is close to $c(r-\alpha)\pi+\theta_0$. This slowing occurs at different values of θ for different values of r, as in the illustrative example above. For our choice of parameters, the slowdown occurs between $\theta=0.7\pi$ for an orbit at the inner edge of the attractor and $\theta=1.12\pi$ for an orbit at the outer edge of the attractor.

This modification of the Rössler equations is reminiscent of the one introduced in [16] with one crucial difference. The present change of coordinates increases the dependence of the phase dependent sensitivity $\boldsymbol{\nabla}_{\mathbf{x}}\theta$ on r without significantly altering the amount of phase diffusion. By contrast, in [16] the Rössler equations were modified so as to increase the amount of phase diffusion significantly. Figure 2 in [16] illustrates that increasing phase diffusion makes phase synchronization more difficult. We illustrate how phase synchronization is similarly affected in the present case.

Since our modification does increase the phase diffusion of system (10) slightly, and our goal is to compare synchronization properties of systems with similar amounts of phase diffusion, we use the unmodified Rössler system with $b=0.16$ for comparison. The phase diffusion coefficient can be estimated as the slope of the variance $\langle(\phi(n)-\langle\phi(n)\rangle)^2\rangle$ [16,4]. The results are given in Fig. 5 and show that the slope of the variance as a function of the number of cycles is 2.04×10^{-3} in the first and 2.37×10^{-3} in the second case. We also compare the variance of $\theta((n+1)T)-\theta(nT)$, where T is the average period of oscillation and find a variance of 0.210 in the first and 0.303 in the second case. Thus the phase diffusion is stronger in the unmodified Rössler system with $b=0.16$.

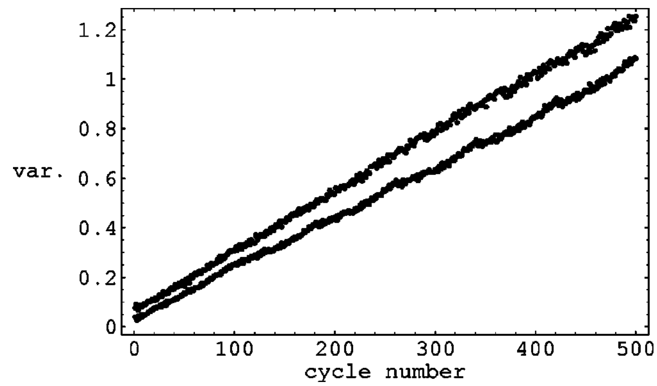

FIG. 5. The time evolution of the variance $\langle(\phi(n)-\langle\phi(n)\rangle)^2\rangle$. The upper and lower line corresponds to the unmodified Rössler system with $b=0.16$ (slope of best linear fit 2.37×10^{-3}), and the modified Rössler system with $b=0.14$ (slope of best linear fit 2.04×10^{-3}), respectively.

Despite the fact that the phase diffusion is smaller for the modified Rössler system, it is more difficult to phase lock, as shown in Fig. 6. Moreover, unlike the regular Rössler equations, infrequent phase slips can still be observed for very strong coupling values.

Lastly we demonstrate that the precision of the synchronization is very different in the two cases. As argued in Sec. II, an increase in the coupling strength will lead to a decrease in the size of the synchronization wedge W if the term $D_V(\mathbf{R},\phi)$ does not play a significant role. Since the size of W determines the amount by which the phase difference $|\phi-\phi_d|$ between the drive and response varies, we expect tighter phase locking with an increase in ϵ (see Fig. 2). However, if $D_V(\mathbf{R},\phi)$ cannot be ignored its the influence increases with the coupling strength ϵ, so that the set W may not shrink, or may even become larger. This is demonstrated in Fig. 7 in which the standard deviation of $\theta(nT_d)$ in the phase locked region are compared. The results show that the difference $|\phi-\phi_d|$ stays large regardless of the value of ϵ in the case of the modified Rössler equations. Repeating the simulations with different parameter values and different types of periodic drives yields similar results.

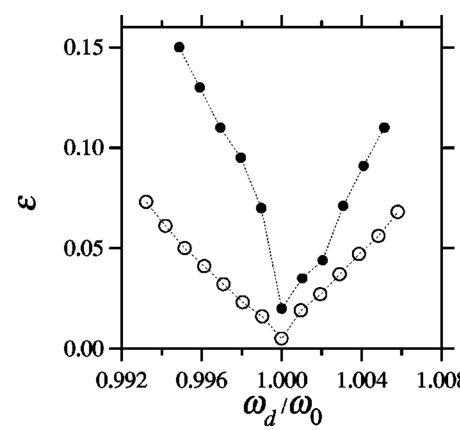

FIG. 6. The solid and open dots form the boundaries of the 1:1 Arnold tongues for the modified Rössler system with $b=0.14$ and the unmodified Rössler system with $b=0.16$, respectively.

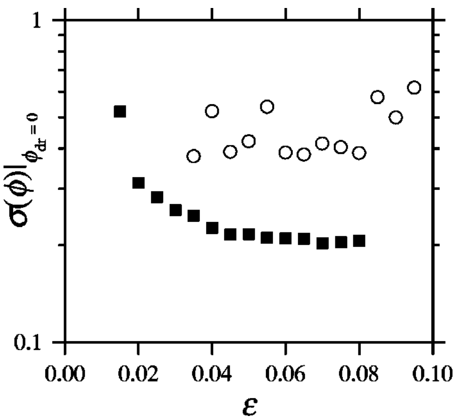

FIG. 7. The variance of the distribution $\theta(nT_d)$ in the region of phase locking for $\omega=\omega_d$. The rectangles and open dots represent data from the unmodified and modified Rössler system respectively. The stronger dependence of the phase dependent sensitivity on **R** in the case of the modified Rössler system leads to less precise phase locking at all coupling values.

This observation has significant consequences. In systems in which the term $D_V(\mathbf{R},\phi)$ is significant, phase locking may occur for sufficiently strong coupling values, however the phase difference between the drive and response may still vary significantly. Moreover, an increase in the coupling strength may not decrease the variation in the phase difference. Since the phase difference may vary by a large amount, phase synchronization may be impossible to detect in such systems. This type of phase synchronization may also not be adequate in systems in which precise timing is necessary, such as neural systems.

VI. CONCLUSION

Among different types of chaotic synchronization [41,42] CPS is of particular interest since it occurs at coupling strengths that are considerably smaller than those necessary for complete synchronization. Because the phase corresponds to a nearly neutral direction within the attractor, under certain conditions only a small driving force is required to control and entrain it. The dynamics in the radial directions can be far more unstable and therefore more difficult to control and synchronize. Chaotic phase-coherent systems can exhibit a richness of behavior while their phase dynamics is still relatively tame, a property with important implications for biological and other systems [6].

We have shown that the ideas used to study phase locking of periodic oscillators can be extended directly and naturally to the chaotic case. This approach provides a way of predicting how a phase-coherent system will phase lock to a periodic driving signal. Systems (10) and (11) were used as illustrative examples because they are most commonly encountered in the literature on CPS, and the change of coordinates necessary to compute $\Gamma(\Psi)$ can be found in a straightforward manner. In the Appendix we show that the change of coordinates required in Sec. II exists for most phase-coherent systems. Therefore these ideas can be applied more generally, although it may be necessary to employ numerical estimates to find an optimal coordinate change. In particular, systems for which several phase variables can be defined may be also considered using the same techniques [24].

The main difference between CPS and phase locking in periodic systems is that the phase sensitivity cannot be assumed to be a function of the phase only. The stronger the dependence on other variables, the less the system will behave like a periodic system when driven by a periodic signal. As shown in Sec. V this means that it may not be possible to synchronize some phase-coherent systems, even if they exhibit very small phase diffusion. Moreover, even if phase synchronization is possible, in such cases the phase difference between the drive and response may vary by an arbitrary amount regardless of the amplitude of the drive.

Let us also note that this view of chaotic synchronization is related to the analysis of randomly or chaotically driven periodic oscillators [39,43]. We may think of the term $\delta(\mathbf{R},\phi)$ as arising from the chaotic or random part of a signal acting on a periodic oscillator since the two situations are equivalent from a mathematical perspective. In cases where $\delta(\mathbf{R},\phi)$ varies slowly compared to ϕ, the adiabatic approximations used in [39] still apply. Following the arguments given in [20] it is also straightforward to extend this approach to the case of coupled chaotic systems. This case will be examined further elsewhere.

In view of these arguments, we expect that our approach has applications beyond CPS. If there exists a change of coordinates in the neighborhood of a chaotic attractor such that in these coordinates certain directions are nearly neutral, we expect the system to be more malleable along these directions. Thus some coordinates may be easier to synchronize than others and partial synchrony of such variables may be achieved before full synchronization of the system occurs.

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APPENDIX A: THE R, ϕ COORDINATE CHANGE

We provide sufficient conditions for the existence of coordinates **R** and ϕ in which the equations of motion take the form (4) and (5).

Theorem 1. Let the system $\mathbf{X}'=f(\mathbf{X})$ have a compact attractor A on which a T-periodic phase coordinate Φ is defined, and assume that f is differentiable. Moreover, assume that the return time $T_{\mathbf{R}_0}$ to the section $\Phi=0$ of any point $(\mathbf{R},\Phi)=(\mathbf{R}_0,0)$ on A satisfies

$$|T-T_{\mathbf{R}_0}|<\eta\ll 1 \tag{A1}$$

and that $\Phi'>0$. Then for any $\epsilon>0$ there exist a coordinate change $\Phi\to\phi$ in a neighborhood of A such that ϕ is T periodic and

$$\phi'=1+\delta(\mathbf{R},\phi),$$

where $|\delta|<\eta+\epsilon$, except for ϕ in a set of measure less than η.

Note that the different approaches for defining the phase of a chaotic system [4,24,25] all result in a phase that is increasing in time. Theorem (A1) states that there exists a change of coordinates to ϕ for any such phase, and thus, up to discrepancies of size at most η, all these definitions are equivalent, as long as there are no resonant modes.

Before we give an outline of the proof it is instructive to consider the case of a system with a limit cycle with period T. We can always define a phase Φ that satisfies

$$\Phi' = 1 + D(\Phi),$$

where Φ is T periodic and $D(\Phi)>-1$ and $\int_0^T D(\Phi)d\Phi = 0$. We can introduce a new phase coordinate

$$\phi = \Phi - \int_0^{\Phi} \frac{D(s)}{1+D(s)}\,ds,$$

where ϕ is also T periodic. A direct computation shows that $\phi'=1$. This change of coordinates stretches the parts in which the phase moves quickly and compresses the parts where it moves slowly, so that its motion becomes uniform.

The proof of the general case follows the same idea. Due to Eq. (A1) and the assumption that $\Phi'>0$ we can write $\Phi' = 1 + D(\mathbf{R},\Phi)$ where $D(\mathbf{R},\Phi)<1$ and

$$\int_0^T D(\mathbf{R}(t),\Phi(t))dt < \eta \qquad \text{(A2)}$$

along any orbit in a neighborhood of A. Moreover, we can express the $\mathbf{R}(t)$ part of a solution with initial condition $(\mathbf{R}(0),\Phi(0)) = (\mathbf{R}_0,0)$ as a function of Φ, i.e, we can define $\mathbf{R}(\Phi,\mathbf{R}_0)$ uniquely for $\Phi\in[0,T)$ and $\mathbf{R}_0$ in a neighborhood of $A\cap\{\Phi=0\}$.

For any orbit in a neighborhood of A the function $D(\mathbf{R}(\Phi,\mathbf{R}_0),\Phi) = D(\mathbf{R}_0,\Phi)$ can be approximated by $\hat{D}(\mathbf{R}_0,\Phi)$ so that $\int_0^T \hat{D}(\mathbf{R}_0,\Phi)d\Phi = 0$ and if $\mathbf{R}_1$ is the $\mathbf{R}$ coordinate of the first return to the section $\Phi=0$ of the orbit starting at $(\mathbf{R}_0,0)$ then $\hat{D}$ satisfies $\hat{D}(\mathbf{R}_0,T) = \hat{D}(\mathbf{R}_1,0)$ and is differentiable at $(\mathbf{R}_0,T) = (\mathbf{R}_1,0)$. Due to Eq. (A2) for any given $\epsilon>0$ we can also choose $\hat{D}$ to satisfy

$$\left| D(\mathbf{R}_0,\Phi) - \hat{D}(\mathbf{R}_0,\Phi)\,\frac{1+D(\mathbf{R}_0,\Phi)}{1+\hat{D}(\mathbf{R}_0,\Phi)} \right| < \eta + \epsilon \qquad \text{(A3)}$$

for all $\Phi\in[0,T)$ outside of a set of measure η. Furthermore, by differentiable dependence on initial conditions, the function $\hat{D}$ can be chosen to be differentiable. We can now introduce a new coordinate

$$\phi = \Phi - \int_0^{\Phi} \frac{\hat{D}(\mathbf{R}_0,s)}{1+\hat{D}(\mathbf{R}_0,s)}\,ds.$$

By our choice of $\hat{D}$ this is a smooth change of coordinates and $\phi' = 1 + \delta(\mathbf{R},\Phi)$ where δ is the quantity on the left side of inequality (A3). This proves the assertion.

There are several important properties of this change of coordinates. First, note that it is performed along every period of Φ. In particular, the value of $\eta_{\mathbf{R}_0} = |T - T_{\mathbf{R}_0}|$ determines the size of $\delta(\mathbf{R},\Phi)$ along one period of Φ for the orbit starting at $(\mathbf{R}_0,0)$. Therefore, the distribution of $\eta_{\mathbf{R}_0}$ is a good indication of how $\delta(\mathbf{R},\Phi)$ behaves over many orbits.

It is also worth noting that if $\Phi' = O(1)$ in a neighborhood of A then condition (A1) is equivalent to the existence of a time T such that

$$|\Phi(0) - \Phi(T)| < K_\eta$$

for $K=O(1)$ and all orbits in a neighborhood of A. Thus it is a matter of choice whether to look at the space or time Poincaré section to determine whether the attractor is phase-coherent.

The choice of the approximating function $\hat{D}(\mathbf{R}_0,T)$ is also somewhat arbitrary. For instance it is possible to choose $\hat{D}(\mathbf{R}_0,\Phi) = D(\mathbf{R}_0,\Phi)$ for all Φ outside of a set of small measure. On the other hand we can choose the functions so that $\hat{D}(\mathbf{R}_0,\Phi) - D(\mathbf{R}_0,\Phi) \neq 0$ but remains small along the entire orbit. As discussed in Sec. II this does not have any significant consequences if we only consider the coarse behavior, but a particular change of coordinates might be preferred if a more detailed study is required.

Lastly, let us remark even if the described change of coordinates exists only on part of the attractor, some of the ideas developed in this paper may still be applicable. If the attractor contains a fixed point, as for instance the Lorenz attractor, then condition (A1) cannot be satisfied and a change of coordinates of the type discussed above does not exist. However, it is possible to find such a coordinate change along orbits or portions of orbits that stay uniformly bounded away from the fixed point, which may lead to partial phase coherence and phase locking as in [29].

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